FURTHER RESULTS ON THE STABILITY ANALYSIS FOR LINEAR DISCRETE TIME-DELAY SYSTEMS: LYAPUNOV – ALGEBRAIC APPROACH

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In this paper, the problem of stability analysis is investigated for a particular class linear discrete time-delay systems. The criteria are derived by Lyapunov’s direct method using minimization with respect to a scalar parameter and they are expressed by norm and eigenvalue of appropriate matrices. The derived results present new delay independent stability conditions for linear discrete time delay systems which can be numerically checked very efficiently.

Key words: Discrete time systems, Time-delay systems, Asymptotic Stability, Lyapunov stability.

INTRODUCTION

Time delay is one of the instability sources for dynamical systems, and is a common phenomenon in many industrial and engineering systems such as those in communication networks, manufacturing, and biology. Since the system stability is an essential requirement in many applications, much effort has been made to investigate the stability criteria for various time-delay systems during the last two decades. For recent progress, refer to [1-10] and the references therein.

Since most physical systems evolve in a continuous time, it is natural that theories for the stability analysis are mainly developed for continuous-time. However, it is more reasonable that one should use a discrete-time approach for that purpose because the
controller is usually implemented digitally. Despite this significance mentioned, less attention has been paid to discrete-time systems with delays [10-20]. It is mainly due to the fact that the discrete-time systems with constant time delays can be transformed into delay-free systems via a state augmentation approach. However, for the systems with either unknown or time-varying delays, this approach cannot be directly applied. Furthermore, for the systems with large known delay amounts, this scheme will lead to large-dimensional systems.

In this note, we revisit the problem of the stability analysis for discrete-time systems with a time delay, which has been investigated in [20]. By using minimization in Lyapunov functional with respect to a scalar parameter and eigenvalues of appropriate matrices, several new results have been obtained for the asymptotic stability. The proposed criteria can be applied in practice in the efficient way.

Throughout this paper we use the following notation. $\mathbb{R}^n$ and $\mathbb{Z}^+$ denote the $n$-dimensional Euclidean space and positive integers. The notation $P > 0$ ($P \geq 0$) means that the matrix $P$ is real symmetric and positive definite (semidefinite). For real symmetric matrices $P$ and $Q$, the notation $P > Q$ ($P \geq Q$) means that matrix $P - Q$ is positive definite (positive semidefinite). $I$ is an identity matrix with appropriate dimension. The superscript “$T$” represents the transpose. $\lambda(P)$, $\rho(P) = \max |\lambda_i(P)|$, $\sigma(P) = \|P\|$ and $\|P\| = \lambda_{\max}^{1/2}(P^TP)$ denote eigenvalue, spectral radius, singular value and Euclidean norm of matrix $P$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

**Problem formulation and some preliminaries**

The dynamic of the system we consider in this article is assumed to be described by the following difference equation:

$$x(k + 1) = A_0x(k) + A_1x(k - h)$$

with an associated function of the initial state

$$x(\theta) = \psi(\theta), \quad \theta \in \{-h, -h + 1, \ldots, 0\}$$

where $x(k) \in \mathbb{R}^n$ is the state at instant $k$, matrices $A_0 \in \mathbb{R}^{n \times n}$ and $A_1 \in \mathbb{R}^{n \times n}$ are constant matrices and $h(k) \in \mathbb{Z}^+$ is the positive integer representing the time delay of the system.

The aim of this article is to establish the sufficient condition that guarantees the delay dependent stability of the class of the system (1). We first introduce the following result, which will be used in the proof of our main results.

**Lemma 1.** For the given matrix $A \in \mathbb{R}^{n \times n}$ and its spectral radius $\rho(A)$ hold

$$\rho(A) < \alpha ,$$

(3)
if and only if for any given matrix \( Q = Q^T > 0 \) there is matrix \( P = P^T > 0 \) such that the following matrix equation is fulfilled
\[
\alpha^2 A^T PA^T - P = -Q .
\] (4)

**Lemma 2.** [20] If for any given matrix \( Q = Q^T > 0 \) there is matrix \( P = P^T > 0 \) such that the following matrix equation is fulfilled
\[
(1 + \varepsilon_m)^T A_0^T P A_0 + (1 + \varepsilon_m^{-1}) A_1^T P A_1 - P = -Q
\] (5)

where
\[
\varepsilon_m = \frac{\|A_1\|_2}{\|A_0\|_2}
\] (6)

then, system (1) is asymptotically stable.

**Remark 1.** In [19] the parameter \( \varepsilon_m \) was given by minimizing a scalar function in Lyapunov functional with respect to a scalar parameter \( \varepsilon \).

**Main results**

**Theorem 1.** System (1) is asymptotically stable, independent of delay if
\[
\|A_1\|_2^2 < \frac{\lambda_{\min}(Q)}{(1 + \varepsilon_m^{-1}) \lambda_{\max}(P)}
\] (7)

where \( P = P^T > 0 \) is the solution of the following Lyapunov discrete matrix equation
\[
(1 + \varepsilon_m)^T A_0^T P A_0 - P = -Q
\] (8)

for any \( Q = Q^T > 0 \).

**Proof.** On the basis of **Lemma 1**, system (1) is asymptotically stable if the following condition is satisfied
\[
(1 + \varepsilon_m)^T A_0^T P A_0 + (1 + \varepsilon_m^{-1}) A_1^T P A_1 - P < 0 .
\] (9)

Further from (8) and (9) we have
\[
(1 + \varepsilon_m^{-1}) A_1^T P A_1 - Q < 0 .
\] (10)

Condition (10) is equivalent to the following condition
\[
\lambda_i \left[ (1 + \varepsilon_m^{-1}) A_i^T P A_i - Q \right] < 0, \quad 1 \leq i \leq n .
\] (11)

Since all matrices in (11) are real and symmetric it follows
\[ \lambda_t \left( (1 + \varepsilon_m^{-1}) A_t^T P A_t - Q \right) \leq \lambda_{\max} \left( (1 + \varepsilon_m^{-1}) A_t^T P A_t - Q \right) \]
\[ \leq \lambda_{\max} \left( (1 + \varepsilon_m^{-1}) A_t^T P \right) - \lambda_{\min} (Q) \]
\[ = (1 + \varepsilon_m^{-1}) \lambda_{\max} \left( A_t^T P^2 P^T A_t \right) - \lambda_{\min} (Q) \]
\[ = (1 + \varepsilon_m^{-1}) \lambda_{\max} \left( \left( P^2 A_t \right)^T P^2 A_t \right) - \lambda_{\min} (Q) \]
\[ = (1 + \varepsilon_m^{-1}) \sigma_{\max}^2 \left( P^2 A_t \right) - \lambda_{\min} (Q) \]
\[ \leq (1 + \varepsilon_m^{-1}) \sigma_{\max}^2 (A_t) \sigma_{\max}^2 \left( P^2 \right) - \lambda_{\min} (Q) \]
\[ = (1 + \varepsilon_m^{-1}) \|A_t\|_2^2 \lambda_{\max} (P) - \lambda_{\min} (Q) < 0 \]

From (12) finally follows (7).

**Lemma 3.** Let for any given matrix \( Q = Q^T > 0 \) there is matrix \( P = P^T > 0 \) being the solution of the following Lyapunov matrix equation
\[ A_0^T P A_0 - P = -Q \]  
then,
\[ (1 + \varepsilon) Q - \varepsilon P > 0, \]
if and only if the following condition is satisfied
\[ \rho(A_0) < \frac{1}{\sqrt{1 + \varepsilon}} . \]

**Proof.** Let \((1 + \varepsilon) Q - \varepsilon P > 0\). Based on (13) following
\[ 0 > \varepsilon P - (1 + \varepsilon) Q = \varepsilon P + (1 + \varepsilon) \left( A_0^T P A_0 - P \right) = (1 + \varepsilon) A_0^T P A_0 - P , \]
i.e.
\[ (1 + \varepsilon) A_0^T P A_0 - P = -R , \]
where \( R = R^T > 0 \). Based on Lemma 1 and (17) we have inequality (15).

**Theorem 2.** Let
\[ \rho(A_0) < \frac{1}{\sqrt{1 + \varepsilon_m}} , \]
and for any given matrix \( Q = Q^T > 0 \) there exists matrix \( P = P^T > 0 \) being the solution of the following Lyapunov matrix equation
\[ A_0^T P A_0 - P = -Q . \]
If the following condition is satisfied
\[ \|A_t\|_2^2 < \frac{\lambda_{\min} \left( (1 + \varepsilon_m) Q - \varepsilon_m P \right)}{(1 + \varepsilon_m^{-1}) \lambda_{\max} (P)} , \]
then the system (1) is asymptotically stable, independent of delay.

**Proof.** From Lemma 2 it follows that the system (1) is asymptotically stable if

\[
(1 + \varepsilon_m) A_0^T P A_0 + (1 + \varepsilon_m^{-1}) A_1^T P A_1 - P =
\]

\[
= (1 + \varepsilon_m^{-1}) A_1^T P A_1 - \left[ P - (1 + \varepsilon_m) A_0^T P A_0 \right]
\]

(20)

Based on Lemma 3, from (18) we have \((1 + \varepsilon_m) Q - \varepsilon_m P > 0\). If we adopt

\[
P - (1 + \varepsilon_m) A_0^T P A_0 = (1 + \varepsilon_m) Q - \varepsilon_m P,
\]

then

\[
(1 + \varepsilon_m) A_0^T P A_0 + (1 + \varepsilon_m^{-1}) A_1^T P A_1 - P
\]

\[
= (1 + \varepsilon_m^{-1}) A_1^T P A_1 - \left[ (1 + \varepsilon_m) Q - \varepsilon_m P \right] < 0
\]

(21)

Condition (21) is equivalent to the following condition

\[
\lambda_i \left\{ (1 + \varepsilon_m^{-1}) A_1^T P A_1 - \left[ (1 + \varepsilon_m) Q - \varepsilon_m P \right] \right\} < 0, \quad 1 \leq i \leq n.
\]

(22)

Since all matrices in (22) are real and symmetric it follows

\[
\lambda_i \left\{ (1 + \varepsilon_m^{-1}) A_1^T P A_1 - \left[ (1 + \varepsilon_m) Q - \varepsilon_m P \right] \right\}
\]

\[
\leq \lambda_{\max} \left( (1 + \varepsilon_m^{-1}) A_1^T P A_1 - \lambda_{\min} \left[ (1 + \varepsilon_m) Q - \varepsilon_m P \right] \right)
\]

\[
= (1 + \varepsilon_m^{-1}) \lambda_{\max} \left( P^2 A_1^T P A_1 - A_1^T P A_1 \right) - \lambda_{\min} \left[ (1 + \varepsilon_m) Q - \varepsilon_m P \right]
\]

\[
= (1 + \varepsilon_m^{-1}) \lambda_{\max} \left( A_1^T P A_1 \right) - A_1^T P A_1 - \lambda_{\min} \left[ (1 + \varepsilon_m) Q - \varepsilon_m P \right]
\]

(23)

\[
\leq (1 + \varepsilon_m^{-1}) \sigma_{\max}^2 \left( P A_1 A_1^T P \right) - \lambda_{\min} \left[ (1 + \varepsilon_m) Q - \varepsilon_m P \right]
\]

\[
= (1 + \varepsilon_m^{-1}) \left\| A_1 \right\|_2^2 \lambda_{\max} \left( P \right) - \lambda_{\min} \left[ (1 + \varepsilon_m) Q - \varepsilon_m P \right]
\]

So, if the condition

\[
(1 + \varepsilon_m^{-1}) \left\| A_1 \right\|_2^2 \lambda_{\max} \left( P \right) - \lambda_{\min} \left[ (1 + \varepsilon_m) Q - \varepsilon_m P \right] < 0
\]

(24)

is satisfied, the system (1) will be asymptotically stable. From (24) it finally follows (19).

**Numerical example**

**Example 1.** Let us consider a discrete delay system described by

\[
x(k + 1) = A_0 x(k) + \delta A_1 x(k - h).
\]
\[ A_0 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & a \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \]

where \( \delta \) is adjustable parameter and system scalar parameter \( a \) takes the following values: -0.15 and 0.5.

The delay-independent asymptotic stability conditions are obtained in terms of \( \delta \) and are summarized in \textit{Table 1}.

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<tr>
<td>-0.15</td>
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<td>Stability margin</td>
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\textbf{CONCLUSION}

In this paper, new sufficient conditions for delay-independent asymptotic stability of linear discrete delay systems are presented based on classical Lyapunov method and using minimization with respect to a scalar parameter. Numerical computations are performed to illustrate the results obtained. It is demonstrated that these results can be applied in practice in the efficient way.

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\textbf{References}


IZVOD

NOVI REZULTATI O STABILNOSTI LINEARNIH DISKRETNIH SISTEMA SA KAŠNJENJEM: LJAPUNOV – ALGEBARSKI PRILAZ

(Originalan naučni rad)

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U ovom radu proučavan je problem stabilnosti posebne klase linearnih diskretnih sistema sa vremenskim kašnjenjem. Kriterijumi stabilnosti su izvedeni pomoću Ljapunove direktnе metode uz korišćenje minimizacije po skalarnom parametru i iskazani su pomoću normi i sopstvenih vrednosti odgovarajućih matrica. Izvedeni rezultati predstavljaju nove uslove stabilnosti za linearne diskretnе sisteme sa kašnjenjem koji su nezavisni od kašnjenja i koji se numerički mogu veoma efikasno proveriti.

Ključne rečи: Diskretnи sistemi, Sistemi sa kašnjenjem, Asimptotska stabilnost, Ljapunovska stabilnost.

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